The interaction between equivalence relations on the symmetric group and pattern avoidance

Talk at Permutation Patterns 2013

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Permutations and patterns

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$$2 \mapsto 4$$
$$3 \mapsto 1$$
$$4 \mapsto 6$$
$$5 \mapsto 3$$
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$$

Patterns are also permutations but we are interested in how they occur in other permutations . . .

Patterns inside permutations

Given a pattern p we say that it occurs in a permutation π if π contains a subsequence that is order-isomorphic to p. If p does not occur in π we say that π avoids the pattern p.

$$Av_n(p) = \text{ permutations in } S_n \text{ that avoid the pattern } p.$$

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Example

Consider the permutation $\pi = 241635$ we had above.

- It has four occurrences of the pattern 123.

$$\underline{24}1\underline{6}35,\ \underline{24}163\underline{5},\ 24\underline{1}6\underline{35},\ \underline{2}416\underline{35}$$

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These are now often called classical patterns.

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Motivation . . . ?

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- More counting sequences: If p is any classical pattern of length 3 then

$$|Av_n(p)| = n\text{-th Catalan number} = \frac{1}{n+1}\binom{2n}{n}.$$

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- Vincular patterns describe Mahonian statistics
- More counting sequences: If p is any classical pattern of length 3 then

$$|Av_n(p)| = n\text{-th Catalan number} = \frac{1}{n+1}\binom{2n}{n}.$$

If we replace p by a vincular pattern of length 3 some more sequences appear, such as the Bell numbers, counting partitions of sets.

Vincular patterns

- They simplify descriptions given in terms of more complicated patterns – factorial Schubert varieties have a very nice description in terms of vincular patterns:

$$X_\pi \text{ smooth if } \pi \text{ avoids } 3412, 4231$$

$$X_\pi \text{ factorial if } \pi \text{ avoids } 3\underline{41}2, 4231$$

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- But compatibility with the symmetry $\pi \mapsto \pi^{\mathrm{i}}$ is no longer valid …

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For any classical pattern

$$|Av_n(p)| = |Av_n(p^{\mathrm{i}})|.$$

But for a vincular pattern this is no longer true in general.
To fix this we need a more general type of pattern.

Bivincular patterns

Bousquet-Mélou, Claesson, Dukes, and Kitaev (2010) defined bivincular patterns as vincular patterns with extra restrictions on the values in an occurrence.

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- It has one occurrence of the pattern $\begin{smallmatrix} 1\,2\,\overline{3}\rceil \\ 1\,2\,3 \end{smallmatrix}$: $\underline{\underline{24}}1\underline{\underline{6}}35$.

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- It has one occurrence of the pattern $\begin{smallmatrix}\lceil 1\,2\,3\\1\,\underline{2\,3}\rfloor\end{smallmatrix}$: $24\underline{\underline{1}}6\underline{\underline{35}}$.

Bivincular patterns

We have now recovered

$$|Av_n(p)| = |Av_n(p^{\mathrm{i}})|, \qquad p \text{ bivincular.}$$

Definitions

Equivalence relations and pattern avoidance

We will now see how equivalence relations interact with pattern avoidance . . .

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... there are many equivalence relations on permutations that one can look at, but we only have time to look at three, so I direct you to http://arxiv.org/abs/1005.5419 if you want to read about some more.

The avoiding classes

Given a bivincular pattern p we are interested in the permutations whose entire equivalence class avoids the pattern

$$\widetilde{Av}_n(p) = \{\pi \in S_n : \pi \text{ and every equivalent permutation avoids } p\}.$$

Definitions

... from Combinatorics

Equivalence relations and pattern avoidance

Equivalence relations on the symmetric group

Conjugacy

Knuth equivalence

Toric equivalence

Other equivalence relations

Conjugacy

The first equivalence relation we are going to look at is conjugacy. Two permutations are said to be conjugate if they have the same cycle type.

Cycle type of a permutation

Given a permutation $\pi \in S_n$ we can write it as a product of disjoint cycles. The cycle type is the partition consisting of the lengths of the cycles

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Example

Consider the permutations in S_3:

π	product of cycles	cycle type
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132		
213		
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231	(123)	[3]
312	(132)	[3]
321	(13)(2)	[2, 1]

Equivalence classes in S_3

So we have three equivalence classes in S_3:

class	elements in class
$[1,1,1]$	123
$[2,1]$	132, 213, 321
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Example

Consider the pattern $p = \begin{smallmatrix}1\\1\end{smallmatrix}$, (1 at the start) and the equivalence classes in S_3:

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$[2, 1]$	$\underline{\underline{1}}32$, 213, 321
$[3]$	231, 312

Example

Consider the pattern $p = \boxed{1}$, (1 at the start) and the equivalence classes in S_3: The class corresponding to cycle type $[3]$ is the only class we count so we get 2.

Permutations without fixed points

If we do this for more n we get

$$|\widetilde{Av}_n\left(\boxed{\begin{matrix}1\\1\end{matrix}}\right)| = 0, 1, 2, 9, 44, 265, 1854, 14833, \ldots, \qquad n = 1, 2, 3, \ldots .$$

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If you look this up on OEIS we find sequence A000166: the subfactorial numbers, counting the number of derangements in S_n, that is, permutations without fixed points.

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$$\widetilde{Av}_n\left(\begin{matrix}\lceil 1 \\ \lfloor 1\end{matrix}\right) = \text{ derangements.}$$

Involutions

For the pattern $\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}$, we get

$$|\widetilde{Av}_n\left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right)| = 1, 2, 4, 10, 26, 76, 232, 764, 2620, \ldots, \qquad n = 1, 2, 3, \ldots .$$

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$$\widetilde{Av}_n\left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right) = \text{ involutions.}$$

Involutions – proof

$$\widetilde{Av}_n\left(\boxed{\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}}\right) = \text{ involutions.}$$

Proof: Take a permutation π that is not in the set on the left. Then some equivalent permutation π' contains the pattern. This means that $\pi' = 23\cdots1\cdots$, so π' has a cycle of length ≥ 3, so π must as well. Therefore π can not be an involution.

Involutions – proof

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Proof: Take a permutation π that is not in the set on the left. Then some equivalent permutation π' contains the pattern. This means that $\pi' = 23\cdots1\cdots$, so π' has a cycle of length ≥ 3, so π must as well. Therefore π can not be an involution.
Now take a permutation π that is not in the set on the right. Then it must have a cycle $(abc\cdots)$ of length ≥ 3. Conjugate π with $(1a)(2b)(3c)$. This gives a permutation, with the same cycle type

$$(1a)(2b)(3c)\pi(1a)(2b)(3c) = \begin{smallmatrix}12\cdots\\23\cdots\end{smallmatrix}$$

that contains the pattern.

This generalizes

It turns out that this pattern is part of a family of patterns:
Let $k \geq 1$

$$\widetilde{Av}_n \left(\begin{array}{|ccccc} \hline 1 & 2 & 3 & \cdots & k \\ 2 & 3 & \cdots & k & 1 \\ \cline{1-4} \end{array} \right) = \text{permutations in of } S_n \text{ only}$$

$$\text{containing cycles of length} < k$$

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Alternatively, two permutations are equivalent if they can be connected through elementary swaps, for example

$$52314 \sim 25314 \sim 25341.$$

Hook-shaped tableaux

The permutations in $\widetilde{Av}_n(231)$ have hook-shaped insertion tableaux, with $1, 2, \ldots k$ in the first line, such as

$$
\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline\end{array}
\qquad
\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}
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It was known that these permutations are the avoiders of $231, 213$, so

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$$\widetilde{Av}_n(231) = Av_n(231, 213) = Av_n(\widetilde{231}).$$

It turns out that this is also true for any classical pattern of length 3, but starts failing for length 4.

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1352046	
2463150	
3504261	
4615302	

Toric classes

Example

We build the toric equivalence class of $\pi = 241635$ as follows:
Place 0 in front of π, then add 1 mod 7 repeatedly:

w/ zero	
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Then we read the list from zero, and that is the equivalence class of π.

Toric classes

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w/ zero	read from zero
0241635	241635
1352046	461352
2463150	
3504261	
4615302	
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Coprime integers

Theorem [U]

For $n \geq 1$

$$|\widetilde{Av}_n\begin{pmatrix}\overline{1\,2}\,\overline{3}| \\ 2\,1\,3\end{pmatrix}| = 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, \ldots$$

Coprime integers

Theorem [U]

For $n \geq 1$

$$|\widetilde{Av}_n\left(\begin{smallmatrix}\overline{1\,2}\,\overline{3}|\\ 2\,1\,3\end{smallmatrix}\right)| = 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, \ldots \qquad \text{(A000010)}$$

Coprime integers

Theorem [U]

For $n \geq 1$

$$
\begin{aligned}
|\widetilde{Av}_n\left(\begin{smallmatrix}\overline{1\,2}\,\overline{3|}\\2\,1\,3\end{smallmatrix}\right)| &= 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, \ldots \qquad \text{(A000010)}\\
&= \phi(n+1),
\end{aligned}
$$

where $\phi(n+1)$ is Euler's totient function, counting the integers that are coprime to $n+1$.

Coprime integers

Theorem [U]

For $n \geq 1$

$$|\widetilde{Av}_n\begin{pmatrix}\overline{1\,2}\,\overline{3}|\\2\,1\,3\end{pmatrix}| = 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, \ldots \qquad \text{(A000010)}$$
$$= \phi(n+1),$$

where $\phi(n+1)$ is Euler's totient function, counting the integers that are coprime to $n+1$.

A crucial step in the proof is that the permutations in the set $\widetilde{Av}_n\begin{pmatrix}\overline{1\,2}\,\overline{3}|\\2\,1\,3\end{pmatrix}$ are the permutations that lie in single element classes.

Divisors

Theorem [U]

For $n \geq 1$

$$|\widetilde{Av}_n \begin{pmatrix} \overline{1\,2}\,3 \\ 2\,1\,3 \end{pmatrix}| = 1, 2, 2, 3, 2, 4, 2, 4, 3, 4, \ldots \qquad \text{(A000005)}$$
$$= d(n),$$

where $d(n)$ counts the number of divisors of n, (sometimes denoted $\sigma_0(n)$).

Example with $n = 6$

Example

$\widetilde{Av}_6\left(\begin{smallmatrix}\overline{1\,2\,3} \\ 2\,1\,3\end{smallmatrix}\right)$		k with $\gcd(k,7)=1$
123456		1
415263		2
531642		3
246135		4
362514		5
654321		6

Example with $n = 6$

Example

$\widetilde{Av}_6\left(\begin{matrix}\overline{1\;2}\;\overline{3}\vert \\ 2\;1\;3\end{matrix}\right)$		k with $\gcd(k,7) = 1$
123456		1
415263		2
531642	bijection?	3
246135	$\longrightarrow$	4
362514		5
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362514	$k =$ location of 1	5
654321		6

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$\widetilde{Av}_6\left(\begin{smallmatrix}\overline{1\,2}\,\overline{3}\\ 2\,1\,3\end{smallmatrix}\right)$		k with $\gcd(k,7)=1$
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531642	bijection	3
246135	$\longrightarrow$	4
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654321		6

The permutations in $\widetilde{Av}_6\left(\begin{smallmatrix}\overline{1\,2}\,3\\ 2\,1\,3\end{smallmatrix}\right)$ are shown in bold.

You can find primes with these sets but . . .

Since $\left|\widetilde{Av}_n\begin{pmatrix}\overline{1\,2}\,3\\2\,1\,3\end{pmatrix}\right|$ gives the number of divisors in n we see that

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$$n \text{ is prime if and only if } \left|\widetilde{Av}_n\begin{pmatrix}\overline{1\,2}\,3\\2\,1\,3\end{pmatrix}\right| = 2,$$

and this gives an extremely inefficient way of checking for primes:

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Since $\left|\widetilde{Av}_n\binom{\overline{1\,2}\,3}{2\,1\,3}\right|$ gives the number of divisors in n we see that

$$n \text{ is prime if and only if } \left|\widetilde{Av}_n\binom{\overline{1\,2}\,3}{2\,1\,3}\right| = 2,$$

and this gives an extremely inefficient way of checking for primes: it takes about 14 seconds for my computer to check that 8 is not a prime.

A conjecture

Let

$$\gamma = \lim_{n \to +\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right)$$

be Euler's constant.

A conjecture

Let

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Conjecture [U]

$$\sum_{\pi \in \widetilde{Av}_n \binom{\overline{1\,2}\,3}{2\,1\,3}} (\text{location of } 1 \text{ in } \pi) < e^{\gamma} \log \log n,$$

is satisfied for all n larger than some constant.

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Conjecture [U]

$$\sum_{\pi \in \widetilde{Av}_n\left(\begin{smallmatrix} \overline{1\,2}\,3 \\ 2\,1\,3 \end{smallmatrix}\right)} (\text{location of } 1 \text{ in } \pi) < e^{\gamma} \log\log n,$$

is satisfied for all n larger than some constant.

This conjecture is equivalent to the Riemann Hypothesis! The sum on the left gives us the sum of the divisors of n, denoted by $\sigma(n)$.

The sum-of-divisors function σ

Theorem [Robin, 1981]

The Riemann Hypothesis is true if and only if

$$\sigma(n) < e^{\gamma} \log \log n,$$

holds for all n larger than some constant.

The largest known violation of this inequality is 5040, which happens to be 7!, the size of S_7, which is a strange coincidence!

Structure of the permutations in $\widetilde{Av}_n\left(\begin{smallmatrix}\overline{1\,2}\,\overline{3}| \\ 2\,1\,3\end{smallmatrix}\right)$

Recall that the permutations in $\widetilde{Av}_n\left(\begin{smallmatrix}\overline{1\,2}\,\overline{3}| \\ 2\,1\,3\end{smallmatrix}\right)$ correspond to the integers d that are coprime to $n+1$. Let me denote them by $\nu_{d,n}$. These permutations turn out to have lots of structure.

Structure of the permutations in $\widetilde{Av}_n\left(\begin{smallmatrix}\overline{1\,2}\,\overline{3}| \\ 2\,1\,3\end{smallmatrix}\right)$

In S_8 we have

$$\nu_{1,8} = 12345678$$
$$\nu_{2,8} = 51627384$$
$$\nu_{4,8} = 75318642$$
$$\nu_{5,8} = 24681357$$
$$\nu_{7,8} = 48372615$$
$$\nu_{8,8} = 87654321.$$

Structure of the permutations in $\widetilde{Av}_n\left(\begin{matrix}\overline{1\,2}\,\overline{3}| \\ 2\,1\,3\end{matrix}\right)$

- They are all constructed the same way.

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In S_8 we have

$$\nu_{1,8} = 12345678$$
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Structure of the permutations in $\widetilde{Av}_n\left(\begin{matrix}\overline{1\,2}\,\overline{3}| \\ 2\,1\,3\end{matrix}\right)$

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In S_8 we have

$$\nu_{1,8} = 12345678$$
$$\nu_{2,8} = 5\underline{\underline{1}}6\underline{\underline{2}}7384$$
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In S_8 we have

$$\nu_{1,8} = 12345678$$
$$\nu_{2,8} = 5\underline{\underline{1}}6\underline{\underline{2}}7\underline{\underline{3}}84$$
$$\nu_{4,8} = 75318642$$
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Structure of the permutations in $\widetilde{Av}_n\left(\begin{matrix}\overline{1\,2}\,\overline{3}| \\ 2\,1\,3\end{matrix}\right)$

- They are all constructed the same way.

In S_8 we have

$$
\begin{aligned}
\nu_{1,8} &= 12345678 \\
\nu_{2,8} &= 5\underline{\underline{1}}6\underline{\underline{2}}7\underline{\underline{3}}8\underline{\underline{4}} \\
\nu_{4,8} &= 75318642 \\
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\end{aligned}
$$

Structure of the permutations in $\widetilde{Av}_n\begin{pmatrix}\overline{1\,2}\,\overline{3}| \\ 2\,1\,3\end{pmatrix}$

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In S_8 we have

$$\nu_{1,8} = 12345678$$
$$\nu_{2,8} = \underline{\underline{51}}6\underline{\underline{2}}7\underline{\underline{3}}8\underline{\underline{4}}$$
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In S_8 we have

$$
\begin{aligned}
\nu_{1,8} &= 12345678\\
\nu_{2,8} &= \underline{\underline{5162}}7\underline{\underline{3}}8\underline{\underline{4}}\\
\nu_{4,8} &= 75318642\\
\nu_{5,8} &= 24681357\\
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Structure of the permutations in $\widetilde{Av}_n\left(\begin{smallmatrix}\overline{12}\,\overline{3}| \\ 2\,1\,3\end{smallmatrix}\right)$

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In S_8 we have

$$\nu_{1,8} = 12345678$$
$$\nu_{2,8} = \underline{\underline{516273}}8\underline{\underline{4}}$$
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Structure of the permutations in $\widetilde{Av}_n \begin{pmatrix} \overline{1\,2}\,\overline{3}| \\ 2\,1\,3 \end{pmatrix}$

- They are all constructed the same way.
- The sum of the first and last elements always equals $n+1$.

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- They are all constructed the same way.
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- If d is a divisor then $\nu_{d,n}$ ends in n/d.

Structure of the permutations in $\widetilde{Av}_n\begin{pmatrix}\overline{1\,2}\,\overline{3}| \\ 2\,1\,3\end{pmatrix}$

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- They are all constructed the same way.
- The sum of the first and last elements always equals $n + 1$.
- If d is a divisor then $\nu_{d,n}$ ends in n/d.
- The increment (difference between two elements) is a constant $\mathrm{mod}(n + 1)$.

Structure of the permutations in $\widetilde{Av}_n\left(\begin{smallmatrix}\overline{1\,2\,3}\\ 2\,1\,3\end{smallmatrix}\right)$

In S_8 we have

$\nu_{1,8} = 12345678$
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- The increment (difference between two elements) is a constant $\mathrm{mod}(n + 1)$.
- They multiply like the numbers they correspond to, for instance, $\nu_{4,8} \circ \nu_{5,8} = \nu_{2,8}$ because $4 \cdot 5 = 20 = 2 \bmod 9$.

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- They multiply like the numbers they correspond to, for instance, $\nu_{4,8} \circ \nu_{5,8} = \nu_{2,8}$ because $4 \cdot 5 = 20 = 2 \bmod 9$.
- If d is a divisor in n then the tableaux that correspond to $\nu_{d,n}$ are box-shaped (and filled in trivially).

Definitions

... from Combinatorics

Equivalence relations and pattern avoidance

Equivalence relations on the symmetric group

Conjugacy

Knuth equivalence

Toric equivalence

Other equivalence relations

Other equivalence relations

- Length of the longest increasing subsequence
- Fixed points
- Shape of the insertion tableau
- Descents
- Major index
- Entropy of a permutation
- Weak excedences
- Number of reduced words
- Signature
- (Number of) saliances
- (Number of) recoils
- Characteristic polynomial of perm. matrix
- Hessenberg form of perm. matrix
- Eigenvalues of perm. matrix
- Number of runs
- *k*-type
- Number of inversions
- Silly-sum
- Number of binary factorizations
- Size of interval to ι in perm. poset
- f-vector of the complex of the above
- Number of anti-chains of interval to ι in perm. poset
- Volume partition
- Superness
- Basic symmetries

. . . and that's the end

Thank you for your time!